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Growth and addition in a herding model

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Abstract. A model of herding is introduced which is exceptionally simple, incorporating only two phenomena, growth and addition. At each time step either (i) with probability p the system grows through the introduction of a new agent or (ii) with probability $q = 1 - p$ a free agent already in the system is added at random to a group of size k with rate A_k. Two versions of the model, $A_k = k$ and $A_k = 1$, are solved and in both versions we find two different types of behaviour. When $p > 1/2$ all the moments of the distribution of group sizes are linear in time for large time and the group distribution is power-law. When $p < 1/2$ the system runs out of free agents in a finite time.

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1 Introduction

There have been a number of models introduced recently [1–5] to model the herding and group kinetics of agents in a market. This herding is believed to account for the fat-tails [6–9] seen in the distribution of returns for a number of financial quantities. These fat-tails are seen on intra-day timescales, and when the distribution of returns is measured on longer timescales, it is found to be Gaussian.

The models can be divided into a number of categories. The first model was introduced by Cont and Bouchaud [1] who considered the network of agents to be a diluted regular lattice and showed how the distribution of the size of the connected clusters on the lattice could lead to a power-law distribution of returns.

Eguíluz and Zimmermann [2] introduced a kinetic version of this model in which groups of agents can either coagulate or fragment at each time step. This model was solved exactly by D'Hulst and Rodgers [3] and since then a number of generalised versions of this model have been introduced [10–12].

In [4,13] a model based on the kinetics of the order book was introduced in which either market or limit bids to buy or sell were made at random. Under simple assumptions about the kinetics of this process, it was shown that this model leads to a power-law distribution of returns.

Finally, in [5] a model of herding in which at each time step either an incoming agent joins an existing group or a group is fragmented into individual agents. The probability of each of these events is fixed. This is a simpler version of a model introduced in [14] in which the above steps occur at each time step, but with rates determined by the number of individual agents.

In this paper we introduce and solve a new model of herding which is simpler than any of those introduced previously. In the next section the model is introduced and solved, in Section 3 the constant coefficient version of the model is considered, and the final section summarises the work.

2 The model

We introduce a model in which at each time step one of two events, chosen at random, can occur. With probability p an agent joins the system but remains free. With probability $q = 1 - p$ an addition event occurs in which a free agent already in the system joins a group of size k with a rate proportional to k. Consequently the number $n_k(t)$ of groups of size $k > 1$ at time t evolves like

$$\frac{\mathrm{d}n_k(t)}{\mathrm{d}t} = \frac{q}{M(t)}\left[(k-1)n_{k-1} - kn_k\right] \tag{1}$$

and the number of groups with only one agent, or equivalently the number of free agents, behaves like

$$\frac{\mathrm{d}n_1(t)}{\mathrm{d}t} = -q\left[\frac{n_1}{M(t)} + 1\right] + p. \tag{2}$$

In these equations

$$N(t) = \sum_{k=1}^{\infty} n_k(t) \tag{3}$$

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represents the number of groups and

$$M(t) = \sum_{k=1}^{\infty} k n_k(t) \tag{4}$$

is the number of agents in the system. The first and second term on the right hand side of equation (1) describe the addition of a new agent to an existing group to, respectively, create and destroy a group of size k. In equation (2) the first term on the right hand side is the destruction of free agents by addition and the second term represents the arrival of free agents. Using rate equations (1, 2) it is a simple matter to show that

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = 2p - 1 \tag{5}$$

and

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t} = p. \tag{6}$$

Equation (5) represents the fact that with probability $2p-1$, *on average*, the number of groups increases by one. Similarly, equation (6) indicates that with probability p the number of agents increases by 1. We can immediately see by looking at equation (5) that, on average, when $p > 1/2$ the system is growing, when $p = 1/2$ the system is static and when $p < 1/2$ the system runs out of monomers in a finite time. We will treat these three cases separately.

Case I: $p > 1/2$

The form of equations (1, 2, 5, 6) suggests that for $p > 1/2$ the solution for $n_k(t)$, for $k = 1, 2, ...$, is linear in time for large t. In this limit we solve equations (5, 6) to yield

$$N(t) = (2p-1)t \qquad \text{and} \qquad M(t) = pt. \tag{7}$$

Writing

$$n_k(t) = t c_k \tag{8}$$

we find that for $k > 1$

$$c_k = \frac{1-p}{p}\left[(k-1)c_{k-1} - k c_k\right]. \tag{9}$$

Using an initial condition obtained from equation (2) we can solve equation (9) to give

$$c_k = p(2p-1)\Gamma\left(1 + \frac{1}{1-p}\right)\frac{\Gamma(k)}{\Gamma(k + \frac{1}{1-p})} \tag{10}$$

where as $k \to \infty$,

$$c_k \sim k^{-\frac{1}{1-p}}. \tag{11}$$

This result can be connected to models of financial markets. If one imagines that at each time step, independent of the group kinetic process, an agent is chosen at random, and the group that agent is in trades by either buying or selling a commodity with equal probability. In this way a group of k agents trades with rate kc_k. If we assume that the traded amount is proportional to the size of the group, then we find that the distribution of returns for the commodity, $R(k)$. This is the equivalent to the distribution of the difference between the number of buyers and sellers, and behaves like

$$R(k) \sim k c_k = p(2p-1)\Gamma\left(1 + \frac{1}{1-p}\right)\frac{\Gamma(k+1)}{\Gamma(k + \frac{1}{1-p})}\cdot \tag{12}$$

In the limit $k \to \infty$ we have power-law behaviour in the distribution of returns

$$R(k) \sim k^{-\beta} \quad \text{with} \quad \beta = \frac{p}{1-p}\cdot \tag{13}$$

We see that β can take any value greater than 1 for $1/2 < p < 1$, with $\beta \to 1$ as $p \to 1/2$ and $\beta \to \infty$ as $p \to 1$.

Case II: $p = 1/2$

When $p = 1/2$ the number of agents in the system does not change, on average. With an initial condition of N free agents, so that

$$N(0) = M(0) = n_1(0) \equiv N, \tag{14}$$

the full time dependent solution is

$$n_k(t) = 2N^2 \frac{t^{k-1}}{[t + 2N]^k}\cdot \tag{15}$$

We thus have that the number of groups of size k grows initially before decaying to zero. The system can run out of free agents and, on average, this occurs when $n_1(t) \sim O(1)$ which occurs on time scales $t \sim O(N^2)$. This observation is in accord with a simple random walk argument; there are N free agents at $t = 0$ and the number increases by 1 with probability $1/2$ and decreases, by either 1 or 2, with probability $1/2$. We would thus expect the time taken to have zero free agents scales as $t \sim O(N^2)$.

From equation (15) have $N(t) = N$ and $M(t) = N + t/2$. In the limit $t \ll N$ the number of groups of size k grows like

$$n_k(t) \sim N\left[\frac{t}{2N}\right]^{k-1}\cdot \tag{16}$$

In the limit $N \ll t \ll N^2$ the group numbers decay as

$$n_k(t) \sim \frac{1}{t} \quad \text{and} \quad M_r(t) \sim \left(\frac{t}{2}\right)^r \tag{17}$$

where $M_r(t)$ is the rth moment, defined by

$$M_r(t) = \sum_{k=1}^{\infty} k^r n_k(t). \tag{18}$$

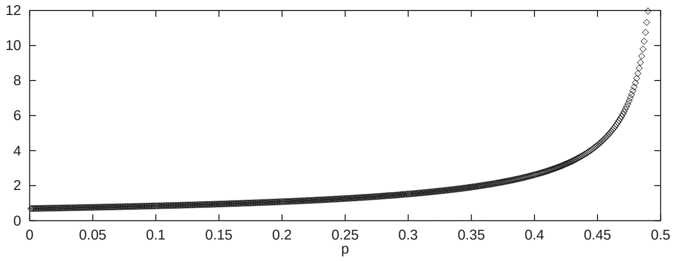

Fig. 1. Average of the time (τ/N) it takes the system to run out of monomers, as a function of p.

Case III: $p < 1/2$

When $p < 1/2$ the number of agents in the system still increases but the number of groups and the number of free agents falls. The system runs out of free agents in a finite time. We can easily solve equation (2) to reveal

$$n_1(t) = N\left[\frac{2(1-p)}{\left[1+\frac{pt}{N}\right]^{\frac{1}{p}-1}} - (1-2p)\left(1+\frac{pt}{N}\right)\right]. \quad (19)$$

At $t = \tau$, $n_1(\tau) = 0$ and τ is given by

$$\tau = \frac{N}{p}\left[\left[\frac{2(1-p)}{1-2p}\right]^p - 1\right]. \quad (20)$$

Thus τ is the average length of time it takes the system to run out of free agents. This shown as a function of p in Figure 1. In the limit when there are no new agents added, $p \to 0$, we have

$$\tau = N \log 2. \quad (21)$$

This time is shorter than N because some of the free agents form a dimer with another monomer, eliminating two free agents in one time step. When $p \to 1/2$ from below the average of the time to run out of free agents diverges as

$$\tau \sim N(1-2p)^{-\frac{1}{2}}. \quad (22)$$

The number of dimers, trimers etc. as well as the total number of agents all become zero in finite times. However, the time taken to run out of free agents is shorter and consequently the most important timescale in the system.

3 Size independent rates

We can introduce a version of the above model in which the rates are independent of the size of a group. In this case the rate equation for the system is

$$\frac{\mathrm{d}n_k(t)}{\mathrm{d}t} = \frac{q}{N(t)}\left[n_{k-1} - n_k\right] \quad (23)$$

for $k > 1$ and $n_1(t)$ obeys

$$\frac{\mathrm{d}n_1(t)}{\mathrm{d}t} = -q\left[\frac{n_1}{N(t)} + 1\right] + p. \quad (24)$$

Using a similar method to that used in the previous section, we find that for $p > 1/2$

$$n_k(t) = (2p-1)^2\frac{(1-p)^{k-1}}{p^k}t \quad (25)$$

and for $p = 1/2$

$$n_k(t) = N\frac{t^{k-1}}{(2N)^{k-1}(k-1)!}\mathrm{e}^{-\frac{t}{2N}}. \quad (26)$$

When $p < 1/2$ the system runs out of free agents in time τ given by

$$\tau = \frac{N}{1-2p}\left[1 - \left[\frac{1-2p}{1-p}\right]^{\frac{1}{p}-2}\right]. \quad (27)$$

As $p \to 0$ we have

$$\tau = \frac{N}{1-\mathrm{e}^{-1}} \quad (28)$$

which, as in the previous model, is less than N because sometimes two free agents are destroyed in one time step. When $p \to 1/2$ from below then τ diverges logarithmically as

$$\tau \sim 2N\log(1-2p) \quad (29)$$

which is slower than the algebraic divergence seen in the linear kernel model.

4 Discussion

A kinetic model for herding has been introduced which incorporates both growth and addition. The system grows as new agents are introduced to the system and the addition mechanism allows groups of agents to be formed. When the growth is fast enough the system exhibits a group size distribution which is power-law with a parameter dependent exponent. When that rate of growth is sufficiently slow the system runs out of free agents in a finite amount of time and the kinetics do not allow the power-law group size distribution to develop. At the boundary between these two cases is a system in which, on average, the number of groups of agents does not change. Here the time it takes the system to run out of free agents, *via* fluctuations about the average behaviour, scales as the square of the initial number of free agents in the system. When the growth is slower than this, the time taken to run out of free agents is proportional to the initial number of free agents in the system.

This behaviour is observed in both the constant coefficient and linear kernel versions of this model.

Models of this type [2,3,5], which are mean-field in character, appear to be well suited to model simple processes in social networks. These include processes in which a property is exchanged between people or in which groups of people are formed who all share the same property. The suitability of these models is in part due to the mean-field, non-local, nature of social interactions in a modern, highly connected world. Processes for which this approach would appear relevant range from information or rumour spreading, particularly in financial markets, through the take up of the latest craze, such as lightweight scooters or Harry Potter, to epidemiological studies of disease and epidemic spread.

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References

1. R. Cont, J.-P. Bouchaud, Macroeconomic Dynamics **4**, 170 (2000)
2. V.M. Eguíluz, M.G. Zimmermann, Phys. Rev. Lett. **85**, 5659 (2000)
3. R. D'Hulst, G.J. Rodgers, Int. J. Theor. Appl. Finance **3**, 609 (2000)
4. S. Maslov, Physica A **278**, 571 (2000)
5. G.J. Rodgers, D. Zheng, Physica A **308**, 375 (2002)
6. B. Mandelbrot, J. Business **36**, 394 (1963)
7. R.N. Mantegna, H.E. Stanley, *An Introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge University Press, Cambridge, 2000)
8. P. Gopikrishnan, M. Meyer, L.A.N. Amaral, H.E. Stanley, Eur. Phys. J. B **3**, 139 (1998)
9. M.M. Dacorogna, U.A. Muller, R.J. Nagler, R.B. Olsen, O.V.Pictet, J. Inter. Money Finance **12**, 413 (1993)
10. R. D'Hulst, G.J. Rodgers, Physica A **280**, 554 (2000)
11. R. D'Hulst, G.J. Rodgers, Eur. Phys. J. B **20**, 619 (2001)
12. D. Zheng, G.J. Rodgers, P.M. Hui, R. D'Hulst, Physica A **303**, 77 (2001)
13. S. Maslov, M. Mills, Physica A **299**, 234 (2001)
14. R. D'Hulst, G.J. Rodgers, Eur. Phys. J. B **21**, 447 (2001)